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Research Article

DYNAMIC MATHEMATICAL MODELING AND CONTROL ALGORITHMS DESIGN OF AN INVERTED PENDULUM SYSTEM

Ayodeji Akinsoji Okubanjo *[1] and Oluwadamilola Kehinde Oyetola [2]

[1] Olabisi Onabanjo University, Engineering, Electrical Electronics and Computer Engineering, Ago-Iwoye, Nigeria
ORCID ID 0000 – 0003 – 1908 – 0365
okubanjo.ayodeji@oouagoiwoye.edu.ng

[2] Olabisi Onabanjo University, Engineering, Electrical Electronics and Computer Engineering, Ago-Iwoye, Nigeria
ORCID ID 0000 – 0001 – 7169 – 6381
oyetola.oluwadamilola@oouagoiwoye.edu.ng

* Corresponding Author

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ABSTRACT

The inimitable features of multivariable, instability, non-minimum phase and non-linearity has established an inverted pendulum system as benchmark to investigate and test new emerging control schemes. In this paper, the objectives are to explicitly model the system dynamics of an inverted pendulum and implement different control algorithms that will stabilize the pendulum in the upright vertical position by controlling the input force applied to the cart in the horizontal position. The mathematical model is derived based on the energy property of Lagrange approach and the control algorithms are expanded on the derived mathematical model in MATLAB-SIMULINK environment. Hence, we proposed four different controls algorithms proportional-integral-derivative controller (PID), pole placement feedback controller (PPFC), linear quadratic regulator controller (LQR) and linear quadratic regulator with estimator (LQR+Estimator) for the control of the linearized inverted pendulum system. The study compares the proposed control algorithms in terms of system response and performance.

Keywords: *Mathematical modeling, Control algorithms, Inverted pendulum, Euler-Lagrange, Simulink*

1. INTRODUCTION

Inverted Pendulum system (IPS) is a classic example of practical model to demonstrate system dynamics and control theory due to its unique features such as multivariable, instability, non-minimum phase and , non-minimum phase and non-linearity. (IRFAN et al., 2013; Ilyas et al., 2017; Krishna et al., 2016; Kumar et al., 2013,). IPS is gaining tremendous attention among the researchers and scholars as a result of its dynamics dominant features that emulates many factual systems in the field of control systems. Also, these dynamics characteristics of inverted pendulum has been established as a baseline to investigate and test new emerging control algorithms. Inverted Pendulum are widely used in balance control of robot manipulator, model flight of rockets and missiles, Segway, unicycle, stabilization of satellite fighting and earthquake resistant of building (Guo and Unversity, n.d.; Kafetzis et al., 2017; Siradjuddin et al., 2017). Recently, IPS are on increasing demand for flying drone especially for balance control of a quadcopter. Inherently, inverted pendulum systems are underactuated mechanical system with complex dynamics which are nonlinear. Instinctively, inverted pendulum possess two equilibrium states i.e. stable state and unstable state (Eizadiyan & Naseriyan, 2015). However, stabilization of the inverted pendulum in an unstable state is a fundamental problem for engineers and scientists. So, several control algorithms have been proposed, implemented and adopted over the past few decades and the quest for new development of inverted pendulum control still continues. Adam and Matlab software based simulation of inverted pendulum is proposed in İlgen *et al.* (2016), a model reference adaptive controller in Krishna *et al*. (2016), Fuzzy control and Genetic Algorithms in Dastranj et al.(2012), PID and LQR in Eizadiyan and Naseriyan, (2015); Jose et al.(2015), fuzzy controller in Sangfeel et al.(2015),state feedback control in Nithya and Vivekanandan,(2014), linear quadratic regulator control, LQR in Chandan et al.(2012), performance comparison model of optimal linear model and Jacobian linearization is proposed in Ababneh et al., 2011, Lagrangian differential transformation approach is proposed in Agarana and Ajayi, (2017), fuzzy and adaptive neuro-fuzzy inference system (ANFIS) in Goswami,(2013),fractional order PID controller in Mishra and Chandra, (2014), Neural network and PID controller in Lee et al.(2009). In the work of Přemysl, Strakoš, Jiří, (2017), a mathematical model of linear inverted pendulum in both state space and transfer function model are derived and pole placement feedback method and a full state observer are implemented. The simulation results justify the superiority of state observer over pole-placement approach. Prayitno et al.(2017), presented a linearized model of an inverted pendulum with three control algorithms, PID, LQR and MPC to stimulate the dynamics of the IPS.

In the research work, the model was analyzed for cases with disturbance and without disturbance and initially controlled with PID controller and later the combined action of PID and LQR, MPC were implemented to compare performances. Hence, it was concluded in the study that the combined action of PID and LQR show better performance over PID alone. Singh and Ph (2015) presented a robust controller to augment the inverted pendulum performance. In their study, a novel H-infinity fuzzy PID controller was proposed and the performance was compared with the conventional PID controller. Simulation results revealed that the new scheme has the affinity to enhance the robustness, transient and steady performance than the PID controller. The use of output feedback controller for inverted pendulum stabilization is addressed in Lee et al.(2015) while pole-placement PI-state feedback controller is designed to stabilize the inverted pendulum cart system in Bettayeb et al. (2014) . Chen et al.(2018) proposed a novel control algorithm to address the repeatability associated with the inverted pendulum when driven by a rotary motion and transmission system. However, Prado et al (2017) analytically and numerically examined the stability of the inverted pendulum based on parametric excitation and large random frequencies. Wang (2011 employed PID controller to address stabilization and tracking problem of three type inverted pendulum and the same problems was addressed via combined action of PD controller and fuzzy PD controllers for a rotational inverted pendulum in Oltean (2014).

In this paper the main objectives are to explicitly model the system dynamics of an inverted pendulum and implement different control algorithms that will stabilize the pendulum in the upright vertical position by controlling the input force applied to the cart in the horizontal position.

Nomenclatures

m_1, m_2	Mass of the cart and the pendulum respectively
k	Spring stiffness coefficients of the cart
b	Friction of the cart
L	Lagrange's function
T	Kinetic energy
V	Potential energy
D	Rayleigh's dissipative function
x	Cart position coordinate
θ	Pendulum angle from vertical
Q_i	Generalized forces
$\boldsymbol{g}$	Center of gravity

Abbreviations

IPS	Degree of freedom
PID	Proportional Integral Derivative Controller
LQR	Linear Quadratic Regulator
PPFC	Pole-Placement Feedback Controller

2. MATHEMATICAL MODELING OF IPS

We derived the dynamics equation of the inverted pendulum using a set of non-linear, second-order, ordinary differential equations and to simulate the dynamics accurately the Lagrangian and Euler-Lagrange was adopted. The motivation for chosen Euler's Lagrange approach over Newton approach is a result of its simplicity, robustness and energy based property. The mathematical model is formulated based on the energy property of Lagrange method and the control algorithms are expounded on the derived mathematical model. However, the Lagrange's equation does not account for dissipative force in the mechanical system, hence, Rayleigh's dissipation function is integrated into Lagrange's equation to form augmented Lagrange's equation. In order to describe the physical motion of the inverted pendulum system, we choose the cart position and pendulum angle as the generalized coordinates. The inverted pendulum shown in fig.1 consists of a cart of mass m_1 and position x, acted upon by a parallel spring-damper configuration with spring stiffness coefficient k and viscous damping coefficient b and the cart suspended a pendulum consisting of a uniform rod of length l, and mass m_2, pivoting about point A. The force U (t) acts on mass of the cart in the direction of x. Subsequently, we derive the differential equations that describes the dynamics of the inverted pendulum using augmented Lagrange's equation in this form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\left(\frac{\partial L}{\partial q_i}\right)+\left(\frac{\partial D}{\partial \dot{q}_i}\right)=\delta_i \tag{1}$$

$$q_i=(x,\theta) \tag{2}$$

The Lagrange function L is defined as the difference of the system's kinetic and potential energy. So, kinetic energy of the IPS as a function of cart and pendulum position and velocity is expressed as:

$$T\left(\theta,\dot{\theta}\right)=\frac{1}{2}\dot{\theta}^{T} M\left(\theta,\dot{\theta}\right)=\frac{m_i v_i^2}{2} \tag{3}$$

Where, $M\left(\theta,\dot{\theta}\right)$ is the nxn inverted pendulum mass matrix and the subscript I denote 1 and 2. Hence, the total kinetic energy of the IPS is the sum of the cart and the pendulum kinetic energies (T_1 and T_2).

$$T\left(\theta,\dot{\theta}\right)=\frac{m_1 v_1^2}{2}+\frac{m_2 v_2^2}{2} \tag{4}$$

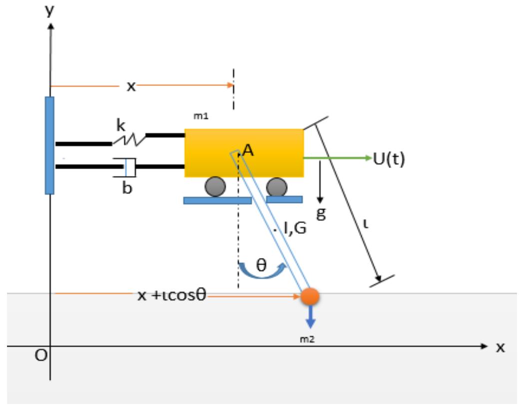

Fig. 1. Inverted Pendulum System (IPS)

To evaluate T_1 and T_2, we need to write the position equations for m_1 and m_2 at point A and subsequently differentiate the respective position to obtain the corresponding velocity and using inner product to obtain the square of the velocity for the cart and pendulum respectively.

$$x_1=x \tag{5}$$

$$y_1=0 \tag{6}$$

$$\begin{aligned} x_2&=x+l\sin(\theta)\\ y_2&=l\cos(\theta)\end{aligned} \tag{7}$$

Let define the velocity as:

$$v=\frac{d}{dt}\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} \dot{x}\\ \dot{y}\end{pmatrix} \tag{8}$$

$$v^2=\|v\|^2=v^T v \tag{9}$$

$$v_1^2=\begin{bmatrix} \dot{x} & 0\end{bmatrix}\begin{bmatrix} \dot{x}\\ 0\end{bmatrix}=\dot{x}^2 \tag{10}$$

Similarly, v_2^2 is computed in the same view

$$\begin{aligned} v_2^2 &=\dot{x}^2+2\dot{x}l\cos(\theta)\dot{\theta}+l^2\cos^2(\theta)\dot{\theta}-\dot{x}l\sin(\theta)\dot{\theta}-\\ &l^2\sin(\theta)\cos(\theta)\dot{\theta}+\dot{x}l\sin(\theta)\dot{\theta}+l^2\sin(\theta)\cos(\theta)\dot{\theta}+\\ &l^2\sin^2(\theta)\dot{\theta}=\dot{x}^2+2\dot{x}l\cos(\theta)\dot{\theta}+l^2\dot{\theta}^2\end{aligned} \tag{11}$$

Substituting v_1^2 and v_2^2 in equation (3), we obtain the kinetic energy of the inverted pendulum as follows:

$$T_1=\frac{1}{2}m_1\dot{x}^2 \tag{12}$$

$$\begin{aligned} T_2&=\frac{1}{2}m_2\left(\dot{x}^2+2\dot{x}l\cos(\theta)\dot{\theta}+l^2\dot{\theta}^2\right)+\\ &\frac{1}{2}I\dot{\theta}^2\end{aligned} \tag{13}$$

So that the total kinetic energy of the inverted pendulum is obtained from equation (12) and (13) and presented as

$$\begin{aligned} T&=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}l^2\dot{\theta}^2+\frac{1}{2}I\dot{\theta}^2+\\ &m_2\dot{x}\dot{\theta}l\cos(\theta)\end{aligned} \tag{14}$$

Reference to the cart level of the IPS (considered as a zero potential), the potential energy of the system is the sum of the potential energies of the cart and pendulum.

$$V = m_2 gl\cos(\theta) + \frac{1}{2}kx^2 \quad (15)$$

The Rayleigh's dissipative function account for damping force in the cart and it is expressed as:

$$D = \frac{1}{2}b\dot{x}^2 \quad (16)$$

Generalized forces:

$$\delta_1 = U(t); \delta_2 = 0$$

The Lagrange formulation defines the behaviour of a dynamic systems in terms of work and energy stored in the system (Urrea & Pascal, 2017). The augmented Lagrange function L is denoted as:

$$L = T - V \quad (17)$$

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}l^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 + m_2\dot{x}\dot{\theta}l\cos(\theta) - \left(m_2 gl\cos(\theta) + \frac{1}{2}kx^2\right) \quad (18)$$

We evaluate the following partial derivatives based on equation (16) and (18) and using chain rule as:

$$\left(\frac{\partial L}{\partial x}\right) = -kx \quad (19)$$

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1 + m_2)\dot{x} + m_2 l\cos(\theta)\dot{\theta} \quad (20)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1 + m_2)\ddot{x} + m_2 l\ddot{\theta}\cos(\theta) - m_2 l\dot{\theta}^2\sin(\theta) \quad (21)$$

$$\left(\frac{\partial L}{\partial \theta}\right) = -m_2 l\dot{x}\sin(\theta)\dot{\theta} - m_2 gl\sin(\theta) \quad (22)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2\ddot{x}l\cos(\theta) - m_2\dot{x}l\sin(\theta)\dot{\theta} + m_2 l^2\ddot{\theta} + I\ddot{\theta} \quad (23)$$

$$\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2\ddot{x}l\cos(\theta) - m_2 xl\sin(\theta)\dot{\theta} + \left[m_2 l^2 + I\right]\ddot{\theta} \quad (24)$$

$$\left(\frac{\partial D}{\partial \dot{x}}\right) = b\dot{x} \quad (25)$$

$$\left(\frac{\partial D}{\partial \dot{\theta}}\right) = 0 \quad (26)$$

For the generalized coordinate x, the Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) + \left(\frac{\partial D}{\partial \dot{x}}\right) = u \quad (27)$$

Substituting the partial derivatives of Eq. (19), (20), (21), (25) into Eq. (27) leads to:

$$(m_1 + m_2)\ddot{x} + m_2 l\cos(\theta)\ddot{\theta} - m_2 l\sin(\theta)\ddot{\theta} + b\dot{x} + kx = u(t) \quad (28)$$

Similarly, for the generalized co-ordinate θ , the Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) + \left(\frac{\partial D}{\partial \dot{\theta}}\right) = u \quad (29)$$

Substituting the partial derivatives of Eq. (22), (23), (24), (26) into Eq. (29) leads to:

$$\left(I + m_2 l_2^2\right)\ddot{\theta} + m_2 l\ddot{x}\cos(\theta) + m_2 gl\sin(\theta) = 0$$

Equation (28) and (29) describe the IPS equations of motion. For simplicity, these equations can be written in terms of inertial matrix, centrifugal force, Coriolis force vector and gravity force in compact matrix form using the generalized coordinate as a column vector θ . Thus, (28)-(29) can be equivalently written as

$$M(\theta)\ddot{\theta} + C\left(\theta, \dot{\theta}\right)\dot{\theta} + G(\theta) = \tau(u)$$

Where the matrices $M(\theta), C\left(\theta,\dot{\theta}\right), G(\theta)$ and $\tau(u)$ are time dependent and

$$M(\theta)=\begin{bmatrix} m_1+m_2 & m_2 l\cos(\theta) \\ m_2 l\cos(\theta) & m_2 l_2^2 + I \end{bmatrix},$$

$$C\left(\theta,\dot{\theta}\right)=\begin{bmatrix} 0 & -m_2\ddot{\theta}^2\sin(\theta)-b\,\dot{x} \\ 0 & 0 \end{bmatrix} \quad (31)$$

$$G(\theta)=\begin{bmatrix} kx \\ m_2 gl\sin(\theta) \end{bmatrix}, \tau(u)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The moment of inertia I has be proven to equal $\frac{m_2 l^2}{12}$ in (Bogdanov, 2004). Apparently, the IPS is strongly nonlinear as a result of the states that exist as product of trigonometric function, however, this function make the system complex in dynamics and challenging to control. Observe now that the inertial matrix $M(\theta)$ is symmetric and nonsingular for every θ, since it determinant is always positive for all θ. So the determinant is

$$\begin{aligned}&\det(M(\theta)) = m_2^2 l^{2^2}\left[1-\cos^2(\theta)\right]+ \\ &(m_1+m_2)I + m_1 m_2 l^2 > 0\end{aligned} \quad (32)$$

To reduce the complexity and to simulate the dynamics accurately, the IPS is linearized around the (upright) equilibrium point such that the system is within the neighborhood of the linear system. This is hold for small deviations in the linear region such that the system state $x=\begin{bmatrix} x & \dot{x} & \theta & \dot{\theta}\end{bmatrix}^T=\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$ and the angle $\theta=\pi+\phi$ is very small such that we can make the approximations

$$\cos(\theta)\approx -1, \sin(\theta)\approx -\phi, \ddot{\theta}\cos(\theta)\approx -\ddot{\phi},$$

$$\dot{\theta}^2\sin(\theta)\approx 0, \dot{\theta}^2 \approx 0$$

Under these approximations equation (28) and (29) become

$$(m_1+m_2)\ddot{x} - m_2 l\ddot{\phi} + b\dot{x} + kx = u \quad (33)$$

$$\left(I+m_2 l^2\right)\ddot{\phi} - m_2 gl\phi = m_2 l\ddot{x} \quad (34)$$

2.1. State Space Representation of the Model

A state space representation is a time domain approach of modeling complex dynamics of single input multiple output and multiple input multiple output systems. However, complex system with many degree-of-freedom and description of such systems with differential equations are often demanding and exhausting. So, state space representation of the system replaces the higher-order differential equations with a first-order matrix differential equations to reduce the system complexity in compact matrix form. Hence, the state and output equations are given in (Norman S, 2011) as:

$$\begin{aligned}\dot{x}(t) &= A..x(t)+B.u(t) \\ y(t) &= C.x(t)+D.u(t)\end{aligned} \quad (35)$$

Where, x, y, u, A, B, C, D are the state vector, output vector, input vector, system matrix, input matrix, output matrix and feedback matrix respectively. Let define the column vector as the state variables

$$x=\begin{bmatrix} x_1 & x_2 & x_3 & x_4\end{bmatrix}^T=\begin{bmatrix} x & \dot{x} & \theta & \dot{\theta}\end{bmatrix}^T \quad (36)$$

Such that
$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x}_1 = \dot{x} \\ x_3 &= \theta \\ x_4 &= \dot{x}_3 = \dot{\theta}\end{aligned}\right\}$$

Hence, equation (33) and (34) can be written in state space representation form as

$$\begin{aligned}
\dot{x}_1 &= x \\
\dot{x}_2 &= \frac{-k\left(I+m_2 l^2\right)}{I(m_1+m_2)+m_1 m_2 l^2}x_1 - \frac{b\left(I+m_2 l^2\right)}{I(m_1+m_2)+m_1 m_2 l^2}x_2 \\
&+ \frac{m_2^2 gl^2}{I(m_1+m_2)+m_1 m_2 l^2}x_3 + \frac{I+m_2 l^2}{I(m_1+m_2)+m_1 m_2 l^2}u \\
\dot{x}_3 &= \theta \\
\dot{x}_4 &= \frac{-m_2 lk}{I(m_1+m_2)+m_1 m_2 l^2}x_1 - \frac{blm_2}{I(m_1+m_2)+m_1 m_2 l^2}x_2 \\
&+ \frac{m_2 gl(m_1+m_2)}{I(m_1+m_2)+m_1 m_2 l^2}x_3 + \frac{m_2 l}{I(m_1+m_2)+m_1 m_2 l^2}u
\end{aligned} \quad (37)$$

Equivalently, equation (37) is presented in terms of A, B, C and D matrices of equation (35) as

$$A=\begin{bmatrix} 0 & 1 & 0 & 1 \\ \frac{-k\left(I+m_2 l^2\right)}{I\left(m_1+m_2\right)+m_1 m_2 l^2} & \frac{-b\left(I+m_2 l^2\right)}{I\left(m_1+m_2\right)+m_1 m_2 l^2} & \frac{m_2^2 g l^2}{I\left(m_1+m_2\right)+m_1 m_2 l^2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-m_2 k}{I\left(m_1+m_2\right)+m_1 m_2 l^2} & \frac{-b l m_2}{I\left(m_1+m_2\right)+m_1 m_2 l^2} & \frac{m_2 g l\left(m_1+m_2\right)}{I\left(m_1+m_2\right)+m_1 m_2 l^2} & 0 \end{bmatrix},$$

$$B=\begin{bmatrix} 0 \\ \frac{I+m_2 l^2}{I\left(m_1+m_2\right)+m_1 m_2 l^2} \\ 0 \\ \frac{m_2 l}{I\left(m_1+m_2\right)+m_1 m_2 l^2} \end{bmatrix}, C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, D=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (38)$$

3. CONTROL ALGORITHMS FOR IPS

Four different control algorithms are implemented on the IPS to effectively control and compare of the model dynamics performances after linearization around equilibrium point. This includes the proportional-integral-derivative controller (PID), Pole –placement approach, Linear Quadratic regulator (LQR) and LQR with Estimator. Since the last three controllers are designed via state space analysis, the model controllability and observability are the key control requirements for arbitrarily closed loop poles placement and state measurements respectively. However, the stability criteria ensures that all eigenvalues of IPS state matrix have a negative real part.

3.1 PID Controllers for IPS

PID controller is widely used in control and mechatronics applications because of its robustness, simplicity in control configuration and suitability for linear system. Hence, Two PID controllers are implemented to control the system such that when the cart reaches a desired position, the inverted pendulum stabilizes in the upright position. The PID controller algorithm combines the P-action, I-action and D-action to adjust the model error. The time domain description of the PID controller is given as:

$$u(t)=k_p e(t)+k_i\int_0^t e(\tau)d\tau+k_D\frac{de(t)}{dt} \qquad (39)$$

Where u (t) is the control signal, the error signal e (t) is defined as e (t) =r (t)-y (t), and r (t) is the reference input signal. Fig.1 shows the PID controller Simulation model of an IPS and the parameters value are presented in Table 1.

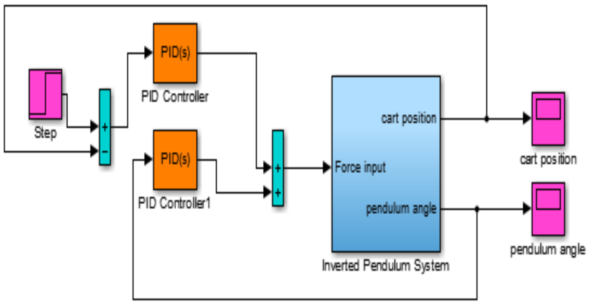

Fig. 2. Simulink implementation of PID controller for IPS

Table 1. parameter of the IPS

Parameter of the IPS	Value	Unit
Mass of the cart m_1	0.5	kg
Mass of the pendulum m_2	0.2	kg
Friction of the cart b	0.1	Ns/m
Spring coefficient of the cart k	0.4	N/m
Length to pendulum center of mass l	0.3	m
Inertial of the pendulum I	0.006	kg-m^2
External force applied to the cart u		
Cart position coordinate x		
Pendulum angle from vertical θ		
Centre of gravity g	9.8	m/s^2

3.2. Pole-Placement Feedback controller (PPFC) for IPS

The dynamics behaviour of the IPS is determined by its closed loop poles position and it is desirable to ensure that the system is fully controllable such that the rank of the controllability matrix $Q_c = \begin{bmatrix} A & AB & A^2B \ldots A^{n-1}B \end{bmatrix}$ is non-zero. Hence, closed loop poles can then be arbitrarily assigned through a static state feedback to the pre-specified position. The control signal u is chosen such that $u = r - kx$ and the closed loop state space model can be written as

$$\dot{x} = Ax + Bu = (A - Bk)x + Br), y = Cx \quad (40)$$

This method depends on the performance criteria such as settling time, steady state error, peak time, maximum overshoot etc. In this design, we want to ensure that the system fulfill less than 5s settling time and overshoot of theta less than 20%. Invariably, the desired characteristic equation of the IPS is formulated from the performance criteria and compared with the IPS closed-loop system's characteristic equation using Ackerman's approach. The dominant close-loop poles of $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ is evaluated from the complex domain specification using the following formulas

$$\xi = \frac{-In\left(\%OS/100\right)}{\sqrt{\pi^2 + In^2\left(\%OS/100\right)}}, T_s < \frac{4}{\xi\omega_n}$$
$$\omega_d = \omega_n\sqrt{1-\xi^2}, \sigma = \xi\omega_n \quad (41)$$

The resulting values from equation (41) are $\xi = 0.707, \omega_n = 9.89rad, \sigma = 7, \omega_d = 7$

So, that the complex dominant poles, $-\sigma \pm j\omega_d$ is approximated as $-7 \pm j7$. The matrices A and B after substituting the parameters value in Table1 turn out to be

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.7273 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ -1.8182 & -0.4545 & 31.1818 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} \quad (42)$$

$$Poles = \left.\begin{matrix} -5.5897 \\ 5.5521 \\ -0.0721 - 0.7543i \\ -0.9721 + 0.7543i \end{matrix}\right\} \quad (43)$$

However, the eigenvalues of A which is the dynamics of the system as depicted in equation (43) are not stable since one of the four poles lies on the right hand side of the s-plane. However, the rank of the controllability matrix confirmed that the system is controllable because the determinant is a non-zero. So, the pole placement approach is computed using a MATLAB function acker () which taken the matrices A, B and poles P as an argument. Where (A, B) is the state space model, P is a vector containing the desired pole positions.

3.3. Linear Quadratic Regulator Controller (LQR) for IPS

It is a known fact that all the desired requirement cannot be satisfied as result of the various tradeoffs that must be made and limitations of the design techniques, hence, optimization based technique that requires some measures of performance index to minimize a cost function is incorporated. We define a cost function depending on the position and the input and minimize it with respect to these parameters, so as to minimize the performance index

$$J = \int_0^\infty \left(x^T Qx + u^T Ru\right)dt \quad (44)$$

Where Q and R, are weight matrices for each parameter and the relative weighting chosen for Q and R determine the relative importance of error reduction and control energy saving. Hence, the controller K, that minimize the cost function J is based on finding the positive definite solution of algebraic Riccati equation (ARE)

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad (45)$$

Such that $u(t) = -kx = -R^{-1}B^T Px(t)$ is optimal for any initial x (0) state. The weighting matrices Q and R are chosen as

$$Q = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R = 1 \quad (46)$$

The matrix Q is selected as indicated in Eq. (46) such that the controller can be easily tuned by changing the non-zero x and y elements in the Q matrix of the m-file function to enhance desirable response. Also, x and y have been used to describe the relative weight of the tracking error in the cart's position and pendulum's angle versus the control effort.

3.4. Linear Quadratic Regulator control with Estimator for IPS

To improve the performance of the IPS, the LQR and the state estimator are combined. The full-order estimator estimates all the state that are not measured. Before we design our estimator, we will first verify that our system is observable. The property of observability determines whether or not based on the measured outputs of the system we can estimate the state of the system. For the system to be completely state observable, the observability matrix $Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ must have rank n where n is the number of state variables of the system. In this case, the system is observable since observability matrix has a maximum of 4. Hence, an m-file function is generated in MATLAB for simulation and the results are discussed extensively.

4. RESULTS AND DISCUSSION

The mathematic dynamics equations of the linearized model is programmed in m-file function of MATLAB and the simulations are run for all the control algorithms. For the PID controller a simulator is built in SIMULINK and the performance of all the control schemes are compared. The control of the inverted pendulum angle is implemented with four different control algorithms namely;

1. Two PID controllers
2. Pole placement feedback controller
3. Linear Quadratic Regulator, LQR and
4. Combination of LQR with Estimator.

Fig 3, 4, 5 and 6 show the graphical step response of the cart's position and the pendulum angle's for all the four control algorithms. In Fig 6 and 7 the cart position and pendulum angle for all the four control schemes are superimposed to facilitate easy performance comparison. In this figure, the responses for the both cart's position and pendulum angles of PID, PPFC, LQR and LQR +Estimator are in cyan, red, blue and magnate respectively. In fig 9 and 10, the cart's position and pendulum angle's step response for all control algorithms are presented for system performance evaluation. Table 2 shows the summary of the performance of the pendulum angle.

Table 2. Pendulum angle simulation results for all control algorithms

Time response specification	PID	PPFC	LQR	LQR+Estimator
Settling Time (Ts)	1.5 s	1.58 s	1.5s	1.48s
Max.Overshoot (%)	10	16	10	9.98
Steady State error	0	0	0	0

It is evident from the Table 2 that the combined action of LQR and Estimator has a settling time of 1.48sec and overshoot of 9.89% which compensate for fast response and stabilize the pendulum angle with minimum overshoot when compared to other algorithms. However, the PID controller and LQR show similar time response characteristics but their overshoot is a lit bit higher when compared with combined action of LQR and Estimator. It can also be deduced that the pole placement feedback controller shows a worst system response. According to the fig 9, it can be deduced that the LQR and Estimator controller exhibit better response and performance. It can be concluded that the combined action of LQR and Estimator is capable of minimizing the error since all the state are available for measurement so as to stabilize the inverted pendulum in the upright position via selecting a weight matrices Q and R that we save control energy and ensure a fair tradeoff between the performance and control effort.

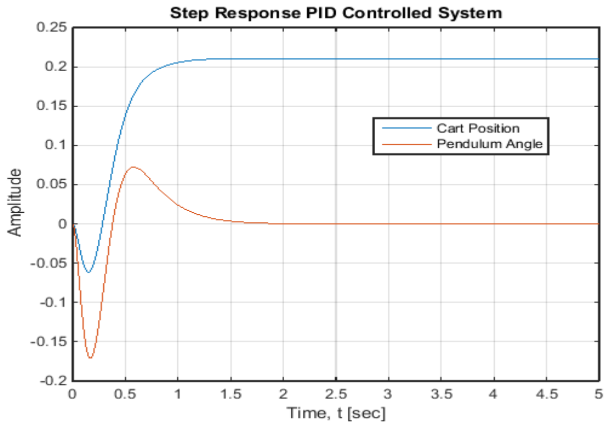

Fig. 3. Step response of PID controller

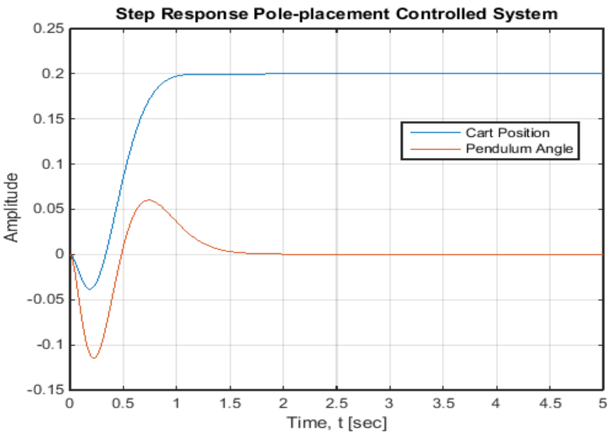

Fig. 4. Step response of PPFC controller

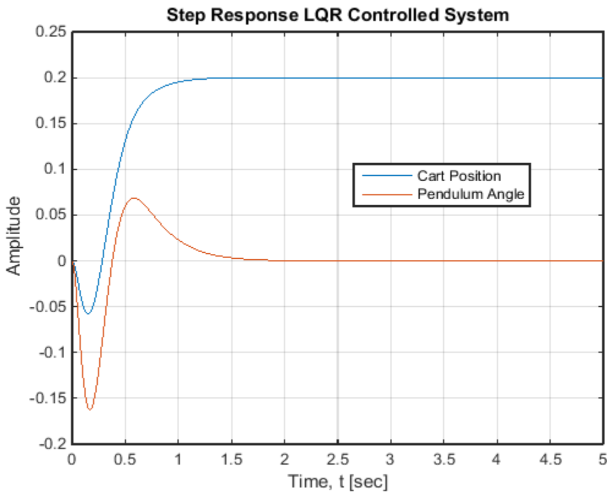

Fig. 5. Step response of LQR controller

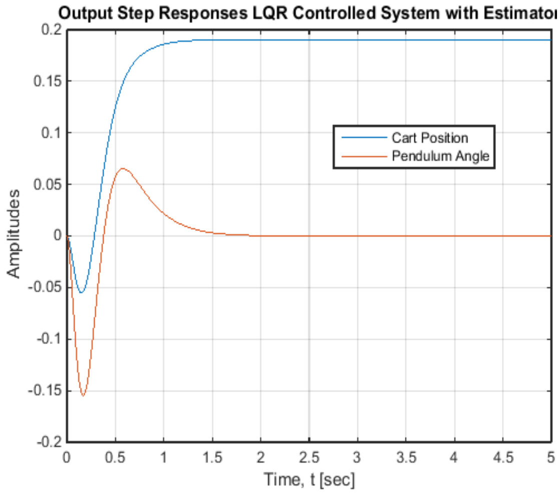

Fig. 6. Step response of LQR + Estimator controller

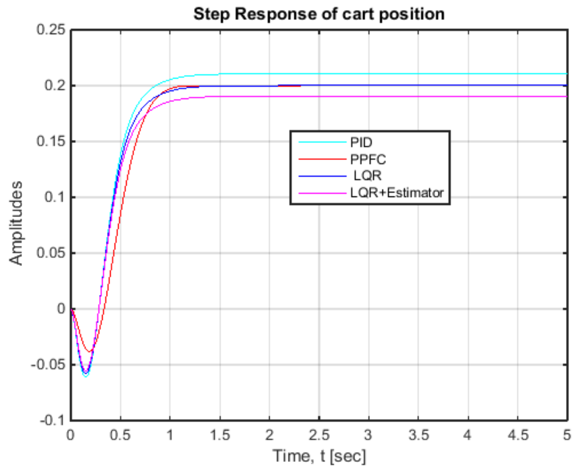

Fig. 9. Step Response for superimposed control algorithms on cart position

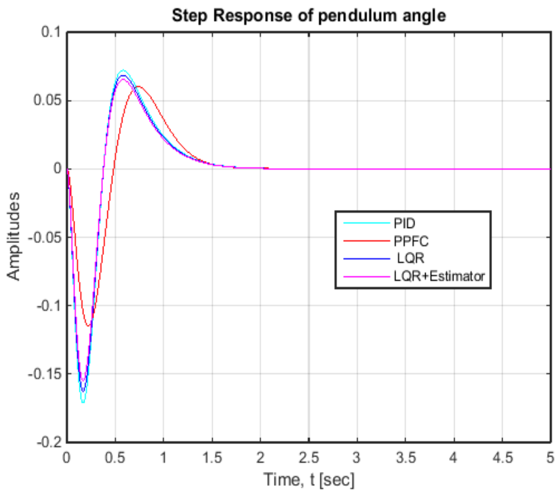

Fig. 10. Step Response for superimposed control algorithms on pendulum angle

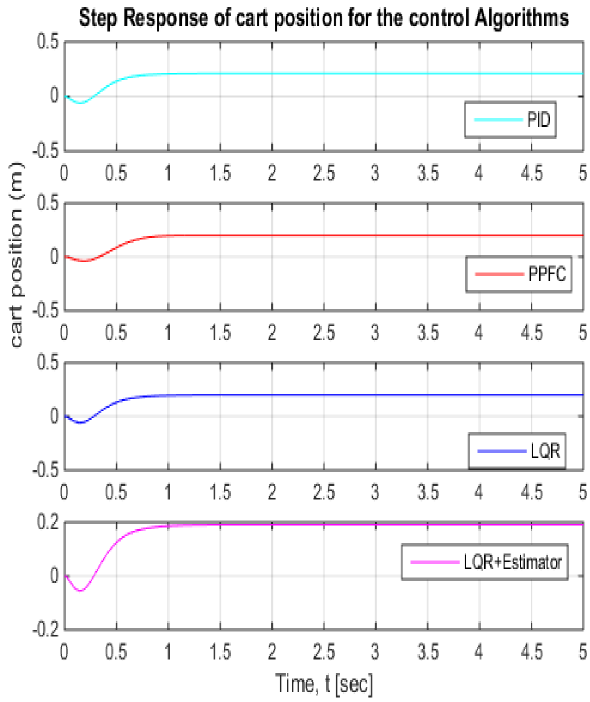

Fig. 7. Step response of cart's position for the control algorithms

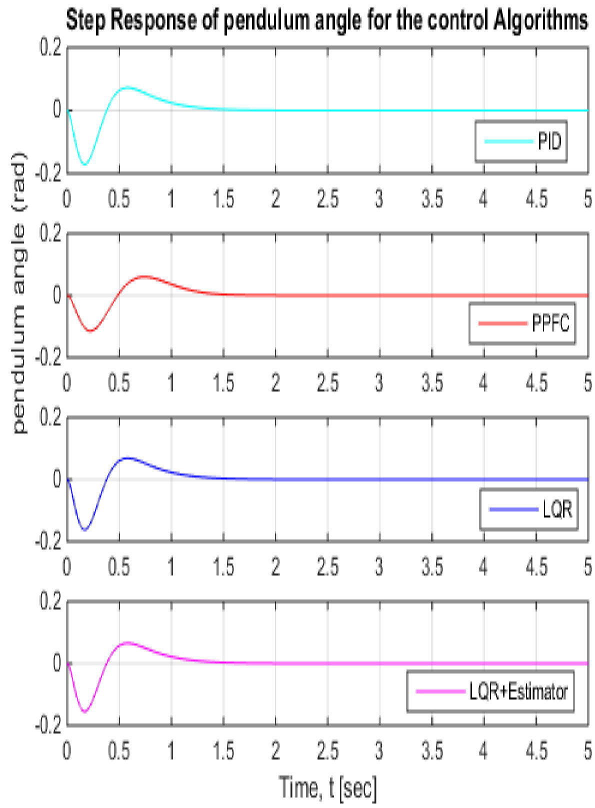

Fig. 8. Step response of pendulum angle's for all the control algorithms

5. CONCLUSION

The dynamic model and control algorithms designed of an inverted pendulum has been successfully formulated and implemented in this paper. The IPS dynamic model was anatomized based on Lagrangian and Euler-Lagrange approach and to simulate the dynamics accurately, the inverted pendulum system is linearized around the upright point such that the system is within the neighborhood of the linear region. Hence, four different control algorithms are implemented with MATLAB/Simulink environment on the linearized model to investigate the performance characteristics of the IPS.

By relating the responses of all the control algorithms of the Table 2, it is found that there is a tradeoff between the response and overshoot of PID as the gain increases or decreases and this significantly influence the PID controller performance. Although, the performance of the of pole-placement feedback controller is higher than other control algorithms as a result of arbitrarily pole location of the poles that require turning for optimal performance at the expense of performance. The response of LQR and LQR+Estimator are similar but it is obvious that LQR +Estimator controllers is a little improved compared to the LQR as it contains an estimator that estimate all the state that are not measured, which in turn contributes to error minimization. Among all the proposed control algorithms, the combined action of LQR+ estimator control scheme gives a better response and performance. This relative performance investigation for this baseline system substantiates that the proposed LQR+Estimator approach is simple, effective and robust for controlling linearized model of dynamic system.

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